

ϕ -HOMOLOGICAL PROPERTIES OF BEURLING ALGEBRAS

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ABSTRACT. In this paper, we investigate ϕ -homological properties for Beurling algebras, where ϕ is a character on those Banach algebras. We show that $L^1(G, w)$ is ϕ_0 -biprojective if and only if G is compact, where ϕ_0 is the augmentation character. Also we show that $M(G, w)$ is $\phi_0^{i_0}$ -biprojective if and only if G is a compact group, where $\phi_0^{i_0}$ is an extension of augmentation character to $M(G, w)$. We define the notion of character-projective Banach A -bimodules and also ϕ -split and ϕ -admissible triples. We show that $L^1(G)$ is amenable if and only if some particular ϕ -admissible triples of Banach $L^1(G)$ -bimodules are ϕ -split triples.

1. INTRODUCTION

Johnson in [5] extended the concept of amenability from locally compact groups to the Banach algebras. In fact A is an amenable (super amenable) Banach algebra, if every continuous derivation $D : A \rightarrow X^*$ ($D : A \rightarrow X$) is inner, for every Banach A -bimodule X , respectively. Also he showed that A is amenable if and only if A has an approximate diagonal, which is, a bounded net $(m_\alpha)_\alpha \subseteq A \otimes_p A$ such that $\pi_A(m_\alpha)a \rightarrow a$ and $a \cdot m_\alpha - m_\alpha \cdot a \rightarrow 0$ for every $a \in A$. In fact, he found out that, there is a direct relation between the Banach algebra $A \otimes_p A$ and the amenability of A . During this period, there exists another approach to the amenable Banach algebras through the homology of Banach algebras. Some concepts like biflat and biprojective Banach algebras were introduced, which they have direct relation with super amenability and amenability, for more details the reader referred to [3] or [8]. Indeed A is a biflat (biprojective) Banach algebra, if there exists a bounded A -bimodule morphism $\rho : A \rightarrow (A \otimes_p A)^{**}$ ($\rho : A \rightarrow A \otimes_p A$) such that $\pi_A^{**} \circ \rho(a) = a(\pi_A \circ \rho(a) = a)$, respectively, see [8]. Recently the notion of left ϕ -amenable and left ϕ -contractible Banach algebras were defined and investigated in [6] and [4]. Indeed A is a left ϕ -amenable (left ϕ -contractible) Banach algebra, if $\mathcal{H}^1(A, X^*) = \{0\}$ ($\mathcal{H}^1(A, X) = \{0\}$), for every Banach A -bimodule X , which its left (right) action defined by $a \cdot x = \phi(a)x$ ($x \cdot a = \phi(a)x$), respectively. The concepts of ϕ -biflat and ϕ -biprojective Banach algebras were introduced by authors in [9]. They showed that $L^1(G)$ is ϕ -biflat if and only if G is an amenable group and $\mathcal{A}(G)$ is ϕ -biprojective if and only if G is a discrete group.

The content of this paper is as follows, after recalling some background notations and definitions, we investigate the ϕ -biprojectivity of Beurling algebras. Then we define a new approach

2000 *Mathematics Subject Classification.* Primary 43A07, 43A20, Secondary 46H05.

Key words and phrases. Beurling algebras, ϕ -biprojective, ϕ -contractible, ϕ -projective module, ϕ -admissible and ϕ -split triple.

to the homology of Banach algebras, which depends on a character of that Banach algebra. We use this approach to show that $L^1(G)$ is amenable if and only if some admissible homological structures of Banach algebras split.

2. PRELIMINARIES

We recall that if X is a Banach A -bimodule, then with the following actions X^* is also a Banach A -bimodule:

$$(a \cdot f)(x) = f(x \cdot a), \quad (f \cdot a)(x) = f(a \cdot x) \quad (a \in A, x \in X, f \in X^*).$$

Let A and B be Banach algebras. The projective tensor product of A with B is denoted by $A \otimes_p B$ and with the following multiplication is a Banach algebra

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = a_1 a_2 \otimes b_1 b_2 \quad (a_1, a_2 \in A, b_1, b_2 \in B).$$

The Banach algebra $A \otimes_p A$ is a Banach A -bimodule with the following actions

$$a \cdot (b \otimes c) = ab \otimes c, \quad (b \otimes c) \cdot a = b \otimes ca \quad (a, b, c \in A).$$

Throughout the paper, $\Delta(A)$ is denoted for the character space of A , that is, all non-zero multiplicative linear functionals on A . Let $\phi \in \Delta(A)$. Then ϕ has a unique extension on A^{**} denoted by $\tilde{\phi}$ and defined by $\tilde{\phi}(F) = F(\phi)$ for every $F \in A^{**}$. Clearly this extension remains to be a character on A^{**} . We denote $\pi_A : A \otimes_p A \rightarrow A$ for the product morphism which is specified by $\pi_A(a \otimes b) = ab$.

Let A be a Banach algebra and X be a Banach A -bimodule. We denote $\mathcal{H}^n(A, X)$ for n^{th} cohomology group of A with coefficients in X . In fact, A is an amenable Banach algebra if and only if $\mathcal{H}^1(A, X^*) = \{0\}$, for every Banach A -module X .

The Banach algebra A is called ϕ -biprojective(ϕ -biflat), if there exists a bounded A -bimodule morphism $\rho : A \rightarrow A \otimes_p A$ ($\rho : A \rightarrow (A \otimes_p A)^{**}$) such that $\phi \circ \pi_A \circ \rho(a) = \phi(a)$ ($\tilde{\phi} \circ \pi_A^{**} \circ \rho(a) = \phi(a)$), respectively, for every $a \in A$. For the further details, we refer readers to [9].

Let G be a locally compact group. A continuous map $w : G \rightarrow \mathbb{R}^+$ is called a weight function, if $w(e) = 1$ and for every x and y in G , $w(xy) \leq w(x)w(y)$ and $w(x) \geq 1$. The Banach algebra of all measurable functions f from G into \mathbb{C} with $\|f\|_w = \int |f(x)|w(x)dx < \infty$ and the convolution product is denoted by $L^1(G, w)$. The Banach algebra of all complex-valued, regular and Borel measures μ on G such that $\|\mu\|_w = \int_G w(x)d|\mu|(x) < \infty$ is denoted by $M(G, w)$. We write $M(G)$, whenever $w = 1$. The map $\phi_0 : L^1(G, w) \rightarrow \mathbb{C}$ which specified by

$$\phi_0(f) = \int_G f(x)dx$$

is called augmentation character and its kernel is called augmentation ideal, for more details see [1].

3. MAIN RESULTS

Let A be a Banach algebra and L be a closed ideal of it. L is left essential as a Banach A -bimodule, if $\overline{AL} = L$.

Let A be a Banach algebra and $\phi \in \Delta(A)$. Suppose that $L \subseteq \ker \phi$ is a closed ideal of A . Clearly ϕ induces a character $\overline{\phi}$ on $\frac{A}{L}$, which is defined by $\overline{\phi}(a + L) = \phi(a)$, for every $a \in A$.

Proposition 3.1. *Let A be a Banach algebra and $\phi \in \Delta(A)$. Suppose that A is a ϕ -biprojective Banach algebra and $L \subseteq \ker \phi$ is a closed ideal of A which is essential as a left Banach A -bimodule. Then $\frac{A}{L}$ is $\overline{\phi}$ -biprojective.*

Proof. Since A is a ϕ -biprojective Banach algebra, there exists a bounded A -bimodule morphism $\rho : A \rightarrow A \otimes_p A$ such that $\phi \circ \pi_A \circ \rho(a) = \phi(a)$ for every $a \in A$. Let $q : A \rightarrow \frac{A}{L}$ be the quotient map. Define $\rho_1 : id \otimes q \circ \rho : A \rightarrow A \otimes_p \frac{A}{L}$. Since L is an essential closed ideal of A , for every $l \in L$, we have

$$\rho_1(l) = id \otimes q \circ \rho(l) = id \otimes q \circ \rho(al') = id \otimes q(\rho(a) \cdot l') = 0,$$

where $l = al'$ for some $a \in A$ and $l' \in L$. Hence there exists an induced map (which still denoted by ρ_1) $\rho_1 : \frac{A}{L} \rightarrow A \otimes_p \frac{A}{L}$. Now define $\rho_2 = q \otimes id_{\frac{A}{L}} \circ \rho_1 : \frac{A}{L} \rightarrow \frac{A}{L} \otimes_p \frac{A}{L}$. We will show that ρ_2 is a bounded $\frac{A}{L}$ -bimodule morphism and $\overline{\phi} \circ \pi_{\frac{A}{L}} \circ \rho_2(x + L) = \overline{\phi}(x + L)$. Suppose that $x \in A$ and $\rho(x) = \sum_{i=1}^{\infty} a_i^x \otimes b_i^x$ for some sequences $(a_i^x)_i$ and $(b_i^x)_i$ in A . Then $\rho_2(x + L) = \sum_{i=1}^{\infty} a_i^x + L \otimes b_i^x + L$, so $\pi_{\frac{A}{L}} \circ \rho_2(x + L) = \sum_{i=1}^{\infty} a_i^x b_i^x + L$. Therefore

$$\overline{\phi}(\sum_{i=1}^{\infty} a_i^x b_i^x + L) = \phi(\sum_{i=1}^{\infty} a_i^x b_i^x) = \phi \circ \pi_A \circ \rho(x) = \phi(x) = \overline{\phi}(x + L).$$

Now suppose that $a + L$ is an arbitrary element of $\frac{A}{L}$. Then $a + L \cdot \rho_2(x + L) = \sum_{i=1}^{\infty} aa_i^x + L \otimes b_i^x + L$. Since ρ is a left A -module morphism, ρ_1 is a left A -module morphism. Hence

$$\begin{aligned} \rho_2(ax + L) &= q \otimes id_{\frac{A}{L}} \circ \rho_1(ax + L) \\ &= q \otimes id_{\frac{A}{L}}(a \cdot \rho_1(x + L)) \\ &= q \otimes id_{\frac{A}{L}}(\sum_{i=1}^{\infty} aa_i^x \otimes b_i^x + L) \\ &= \sum_{i=1}^{\infty} aa_i^x + L \otimes b_i^x + L \\ &= a + L \cdot \sum_{i=1}^{\infty} a_i^x + L \otimes b_i^x + L \\ &= a + L \cdot \rho_2(x + L). \end{aligned}$$

Similarly one can show that ρ_2 is a right $\frac{A}{L}$ -module morphism and the proof is complete. \square

Now we are going to define some homological properties for some particular Banach modules. In fact the following definition is motivated by the concepts of Banach homology, for more details about the parallel concepts the reader referred to [8, Section 5].

Definition 3.2. A Banach A -bimodule X is called left ψ -projective, if there exists a non-zero character ψ on X and a bounded left A -module morphism $\rho : X \rightarrow A \otimes_p X$ such that $\psi \circ \pi_{A,X} \circ \rho(x) = \psi(x)$, where $\pi_{A,X} : A \otimes_p X \rightarrow X$ specified by $\pi_{A,X}(a \otimes x) = a \cdot x$. Similarly if X is a Banach A -bimodule and ρ is a right A -module morphism, then we can define the right ψ -projective Banach A -bimodules. A Banach A -bimodule X is called ψ -projective, if there exist bounded A -bimodule morphisms $\rho : X \rightarrow A \otimes_p X$ and $\xi : X \rightarrow X \otimes_p A$ such that $\psi \circ \pi_{A,X} \circ \rho(x) = \psi(x)$ and $\psi \circ \pi_{X,A} \circ \xi(x) = \psi(x)$, where ψ is a character on X .

Note that a Banach algebra A is ϕ -biprojective if and only if the Banach A -module $X = A$ with natural actions is ϕ -projective.

Using the same argument as in the [10, Remark 4.9] we will give an example which shows the differences of the projective Banach A -bimodules (for the definition see [8, Section 5]) and the ψ -projective Banach A -bimodules.

Example 3.3. Let G be the integer Heisenberg group. One can see that G has an open abelian subgroup but is not a finite extension of an abelian group. We go toward a contradiction and suppose that $X = \mathcal{A}(G)$ is a projective $\mathcal{A}(G)$ -bimodule. Then by [8, Theorem 5.1.10], $\mathcal{A}(G)$ is biprojective, hence is biflat. On the other hand, since G is amenable by Leptin's theorem $\mathcal{A}(G)$ has a bounded approximate identity, therefore $\mathcal{A}(G)$ is an amenable Banach algebra which has contradiction with main result of [2]. By [9, Corollary 1] it is easy to see that $X = \mathcal{A}(G)$ is a ϕ -projective Banach $\mathcal{A}(G)$ -bimodule.

Let G be a discrete group. We want to show that $X = \mathcal{A}(G) \otimes_p \mathcal{A}(G)$ is a ψ -projective Banach $\mathcal{A}(G)$ -bimodule, for some character $\psi \in \Delta(\mathcal{A}(G) \otimes_p \mathcal{A}(G))$. To see this, pick $t_0 \in G$ and denote $\chi_{\{t_0\}}$ for the characteristic function at t_0 . It is well-known that ϕ_{t_0} is a character on $\mathcal{A}(G)$, which specified by $\phi_{t_0}(f) = f(t_0)$, for every $f \in \mathcal{A}(G)$. Consider $\psi = \phi_{t_0} \otimes \phi_{t_0}$, which is a character on $\mathcal{A}(G) \otimes_p \mathcal{A}(G)$ and defined by $\phi_{t_0} \otimes \phi_{t_0}(f \otimes g) = f(t_0)g(t_0)$, where f and g in $\mathcal{A}(G)$. Since for every $f \in \mathcal{A}(G)$ we have $f\chi_{\{t_0\}} = \chi_{\{t_0\}}f = \phi_{t_0}(f)\chi_{\{t_0\}}$ and $\phi_{t_0}(\chi_{\{t_0\}}) = 1$, the map $\rho : X \rightarrow \mathcal{A}(G) \otimes_p X$ defined by $\rho(x) = \psi(x)\chi_{\{t_0\}} \otimes \chi_{\{t_0\}} \otimes \chi_{\{t_0\}}$, is an $\mathcal{A}(G)$ -bimodule morphism. One can easily see that $\psi \circ \pi_{\mathcal{A}(G),X} \circ \rho = \psi$. Similarly we can define ξ and show that $\psi \circ \pi_{\mathcal{A}(G),X} \circ \xi = \psi$, hence X is a ψ -projective Banach $\mathcal{A}(G)$ -bimodule.

Lemma 3.4. *Let A be a ϕ -biprojective Banach algebra with $\phi \in \Delta(A)$. Let $L \subseteq \ker \phi$ be a closed ideal of A which is left essential as a A -bimodule. Then $\frac{A}{L}$ is a left $\bar{\phi}$ -projective Banach A -bimodule.*

Proof. Using the notations as in the proof of the previous Proposition, one can easily see that ρ_1 is a bounded left A -module morphism and $\overline{\phi} \circ \pi_{A, \frac{A}{L}} \circ \rho_1(x + L) = \overline{\phi}(x + L)$. Then the proof is complete. \square

We recall that $m \in A \otimes_p A$ is a ϕ -Johnson contraction for A , if $a \cdot m = m \cdot a$ and $\phi \circ \pi_A(m) = 1$, where $a \in A$, for more details the reader referred to [9].

Let A be a Banach algebra and $\phi \in \Delta(A)$. A is left ϕ -contractible if and only if there exists an element m in A such that $am = \phi(a)m$ and $\phi(m) = 1$. For the further details the reader referred to [4] and [7].

Compare the following proposition with [3, Theorem 5.13].

Proposition 3.5. *Let G be a locally compact group. Then the following statements are equivalent:*

- (i) $L^1(G, w)$ is ϕ_0 -biprojective;
- (ii) $L^1(G, w)$ is left ϕ_0 -contractible;
- (iii) G is compact.

Proof. (i) \Rightarrow (ii) Let $A = L^1(G, w)$ and $L = \ker \phi_0$, where ϕ_0 is the augmentation character and let A be ϕ_0 -biprojective. Since A has a bounded approximate identity, L becomes a left essential Banach A -bimodule. Thus by Lemma 3.4, $\frac{A}{L} \cong \mathbb{C}$ is left $\overline{\phi_0}$ -projective A -bimodule. So there exists a bounded left A -module morphism $\rho : \mathbb{C} \rightarrow A \otimes_p \mathbb{C} \cong A$ such that $\overline{\phi_0} \circ \pi_{A, \frac{A}{L}} \circ \rho(c) = \overline{\phi_0}(c)$, where $c \in \mathbb{C}$. Let $m = \rho(1) \in A$. Then $\phi_0(m) = \phi_0(\rho(1)) = \overline{\phi_0} \circ \pi_{A, \frac{A}{L}} \circ \rho(1) = 1$ and $a \cdot \rho(1) = \rho(a \cdot 1) = \phi_0(a)\rho(1)$, where $a \in A$. In fact since \mathbb{C} with left outer multiplication $a \cdot z = \phi_0(a)z$ is a projective left A -module, for every $a \in A$. Then A is left ϕ_0 -contractible, see [7, Theorem 4.3].

(ii) \Rightarrow (iii) Suppose that A is a left ϕ_0 -contractible Banach algebra. Then there exists an element $m \in A$ such that $am = \phi_0(a)m$ and $\phi_0(m) = 1$, where $a \in A$. Let $g \in G$ be an arbitrary element and $f \in A \setminus L$. Hence

$$\phi_0(f)\delta_g * m = \delta_g * (f * m) = (\delta_g * f) * m = \phi_0(\delta_g * f)m = \phi_0(f)m.$$

Hence m is constant and belongs to A , so is 1. Therefore

$$|G| = \int_G w(e)dx \leq \int_G w(x)dx < \infty,$$

so G is a compact group.

(iii) \Rightarrow (i) Let G be a compact group. Then $m = 1 \otimes 1$ in $A \otimes_p A$ satisfies $a \cdot m = m \cdot a = \phi_0(a)m$ and $\phi_0 \circ \pi_A(m) = 1$, (we use normalized left Haar measure here), where $a \in A$, then A is ϕ_0 -Johnson contractible. Hence [9, Lemma 3.2] finishes the proof. \square

One can easily see that every biprojective Banach algebra A is ϕ -biprojective for every $\phi \in \Delta(A)$ but the converse is not always true, see Example 3.3. At the following corollary, we show that [8, Theorem 5.2.30] for the group algebras, in the ϕ -biprojective case, is also valid. It is well-known that if G is compact, then G is amenable, hence by [8, Theorem 2.4.7] $\mathcal{H}^n(L^1(G), X^*) = \{0\}$, for every $n \in \mathbb{N}$ and for every Banach $L^1(G)$ -bimodule X .

Corollary 3.6. *Let G be a locally compact group.*

- (i) *If $L^1(G)$ is ϕ_0 -biprojective, then for every Banach $L^1(G)$ -bimodule X , $\mathcal{H}^n(L^1(G), X) = \{0\}$, where $n \geq 3$;*
- (ii) *$L^1(G)$ is ϕ_0 -biprojective if and only if $\mathcal{H}^1(L^1(G), X) = \{0\}$, for every Banach $L^1(G)$ -bimodule X with $x \cdot a = \phi_0(a)x$ such that $a \in L^1(G)$ and $x \in X$.*

Proof. Using this fact that $L^1(G)$ is biprojective if and only if G is compact, see [3, Theorem 5.13], and by Proposition 3.5 and [8, Theorem 5.2.30] one can easily show the part (i) is true. Part (ii) is true by Proposition 3.5. \square

Suppose that $\phi \in \Delta(L^1(G, w))$. Pick an element $i_0 \in L^1(G, w)$ such that $\phi(i_0) = 1$. We define $\phi^{i_0} : M(G, w) \rightarrow \mathbb{C}$ by $\phi^{i_0}(a) = \phi(ai_0)$. Since $L^1(G, w)$ is a closed ideal of $M(G, w)$, we have

$$\begin{aligned}
 \phi^{i_0}(ab) &= \phi(ab i_0) = \phi(i_0) \phi(ab i_0) = \phi(i_0 a b i_0) \\
 &= \phi(i_0 a b i_0) \\
 &= \phi(i_0 a) \phi(b i_0) \\
 &= \phi(i_0 a i_0) \phi(b i_0) \\
 &= \phi(a i_0) \phi(b i_0) \\
 &= \phi^{i_0}(a) \phi^{i_0}(b).
 \end{aligned}$$

Then ϕ^{i_0} is a character on $M(G, w)$ such that $\phi^{i_0}|_{L^1(G, w)} = \phi$.

Corollary 3.7. *Let G be a locally compact group and let ϕ_0 be the augmentation character of $L^1(G, w)$. Then $M(G, w)$ is $\phi_0^{i_0}$ -biprojective if and only if G is compact.*

Proof. Let $M(G, w)$ be $\phi_0^{i_0}$ -biprojective. Then, since $\phi_0^{i_0}|_{L^1(G, w)} \neq 0$ and $L^1(G, w)$ is a closed ideal of $M(G, w)$, by [9, Lemma 3.2] $L^1(G, w)$ is ϕ_0 -biprojective. Now by Proposition 3.5, G is a compact group.

For converse, let G be a compact group. Then $1 \otimes 1$ is a ϕ_0 -Johnson contraction for $L^1(G, w)$. By similar arguments as in the proof of [9, Proposition 2.2], one can show that $L^1(G, w)$ is left and right ϕ_0 -contractible. Since $L^1(G, w)$ is a closed ideal in $M(G, w)$ and $\phi_0^{i_0}|_{L^1(G, w)} \neq 0$, one can easily see that $M(G, w)$ is left and right $\phi_0^{i_0}$ -contractible, by using the similar arguments as

in the proof of [9, Proposition 2.2], one can easily see that $M(G, w)$ is $\phi_0^{i_0}$ -Johnson contractible. Now apply [9, Lemma 3.2] to show that $M(G, w)$ is $\phi_0^{i_0}$ -biprojective. \square

Suppose that A is a Banach algebra and $a \in A$. It is well-known that two maps $b \mapsto ab$ and $b \mapsto ba$ are $w^* - w^*$ -continuous on A^{**} , where $b \in A^{**}$.

Recall that A is a ϕ -Johnson amenable Banach algebra for some $\phi \in \Delta(A)$, if there exists an element $m \in (A \otimes_p A)^{**}$ such that $\tilde{\phi} \circ \pi_A^{**}(m) = 1$ and $a \cdot m = m \cdot a$, for every $a \in A$, see [9, Definition 1.1].

Lemma 3.8. *Let G be a locally compact group and let ϕ_0 be the augmentation character of $L^1(G)$. If $M(G)^{**}$ is $\tilde{\phi}_0^{i_0}$ -biprojective, then G is amenable. Converse holds whenever G is a compact group.*

Proof. Let $M(G)^{**}$ be $\tilde{\phi}_0^{i_0}$ -biprojective. Then by [9, Lemma 3.5], $M(G)$ is $\phi_0^{i_0}$ -biflat. Since $M(G)$ has a unit, by [9, Proposition 3.3] $M(G)$ is $\phi_0^{i_0}$ -Johnson amenable, hence by [9, Theorem 2.2] $M(G)$ is left $\phi_0^{i_0}$ -amenable. Since $L^1(G)$ is a closed ideal of $M(G)$ and $\phi_0^{i_0}|_{L^1(G)} \neq 0$, by [6, Lemma 3.1] $L^1(G)$ is left ϕ_0 -amenable, so G is an amenable group, see the proof of [8, Theorem 2.1.8].

Let G be compact. Then it is easy to see that 1 is a left and right ϕ_0 -contraction for $L^1(G)$ where we consider normalized Haar measure. One can easily show that also 1 is a left and right $\phi_0^{i_0}$ -contraction for $M(G)$, since $L^1(G)$ is a closed ideal in $M(G)$. Now by $w^* - w^*$ -continuity of the maps $b \mapsto b1$ and $b \mapsto 1b$, one can easily see that 1 is also a left and right $\tilde{\phi}_0^{i_0}$ -contraction for $M(G)^{**}$. Define $\rho : M(G)^{**} \rightarrow M(G)^{**} \otimes_p M(G)^{**}$ by $\rho(b) = b \cdot 1 \otimes 1$. It is easy to see that ρ is a bounded $M(G)^{**}$ -bimodule morphism and $\tilde{\phi}_0^{i_0} \circ \pi_{M(G)^{**}} \circ \rho(b) = \tilde{\phi}_0^{i_0}(b)$ and the proof is complete. \square

Definition 3.9. Let A be a Banach algebra and $\phi \in \Delta(A)$. Suppose that X is a Banach A -bimodule. A character ψ on X is called a ϕ -character, if $\psi(a \cdot x \cdot b) = \phi(a)\phi(b)\psi(x)$ for every a and b in A and x in X .

Example 3.10. Let A be a Banach algebra and $\phi \in \Delta(A)$. It is well-known that $\phi \otimes \phi \in \Delta(A \otimes_p A)$, where $\phi \otimes \phi(a \otimes b) = \phi(a)\phi(b)$. Clearly via the following module actions $A \otimes_p A$ is a Banach A -bimodule:

$$a \cdot x \otimes y = ax \otimes b, \quad x \otimes y \cdot a = x \otimes ya,$$

for every a, x and y in A . Hence, $\phi \otimes \phi$ is a ϕ -character, since

$$\phi \otimes \phi(a \cdot x \otimes y \cdot b) = \phi \otimes \phi(ax \otimes yb) = \phi(a)\phi(b)\phi \otimes \phi(x \otimes y),$$

for every a, b, y and x in A .

Another example for the notion of ϕ -character is given here. For $\phi \in \Delta(A)$, it is easy to see that, there exists a unique extension of ϕ to A^{**} , denoted by $\tilde{\phi}$, such that $\tilde{\phi}(F) = F(\phi)$, for every $F \in A^{**}$. It is easy to see that $\tilde{\phi} \in \Delta(A^{**})$ and A^{**} is a Banach A -bimodule naturally.

Also for every a, y in A and x in A^{**} , we have $\tilde{\phi}(a \cdot x \cdot y) = \phi(a)\phi(y)\tilde{\phi}(x)$. So $\tilde{\phi}$ is a ϕ -character. The notion of ϕ -character reveals even for the quotient algebras. Let $\phi \in \Delta(A)$ and $L \subseteq \ker \phi$ be a closed ideal of A . Suppose that $\bar{\phi}$ is a induced character on $\frac{A}{L}$ by ϕ . Clearly $\frac{A}{L}$ is a Banach A -bimodule. Also since

$$\bar{\phi}(axy + L) = \phi(axy) = \phi(a)\phi(x)\phi(y) = \phi(a)\phi(y)\bar{\phi}(x + L)$$

for every a, y and x in A , $\bar{\phi}$ is a ϕ -character.

Definition 3.11. Let A be a Banach algebra and $\phi \in \Delta(A)$. Suppose that E and P are Banach A -bimodules. Let the Banach A -module P has a ϕ -character, say ψ , and also let g be a bounded Banach A -module morphism from E into P . The triple (E, P, g) is called ϕ -admissible, if there exists a bounded, linear map $\rho : P \rightarrow E$ such that $\psi \circ g \circ \rho(p) = \psi(p)$, for every $p \in P$, also is called ϕ -split, whenever ρ is a bounded A -bimodule morphism.

Clearly if the triple $(A \otimes_p A, A, \pi_A)$ is ϕ -split, then A is ϕ -projective Banach A -bimodule (with respect to the ϕ -character ϕ), hence A is ϕ -biprojective.

Proposition 3.12. Let A be a ϕ -Johnson amenable Banach algebra, for $\phi \in \Delta(A)$. Then every ϕ -admissible, triple (E^{**}, P^{**}, g) is ϕ -split triple, for every Banach A -bimodules E and P , provided g is a $w^* - w^*$ -continuous A -bimodule morphism.

Proof. Since A is ϕ -Johnson amenable, there exists a bounded net $(m_\alpha)_\alpha \subseteq A \otimes_p A$ such that

$$a \cdot m_\alpha - m_\alpha \cdot a \rightarrow 0, \quad \phi \circ \pi_A(m_\alpha) \rightarrow 1,$$

for every $a \in A$, see [9]. Let define a bounded linear map $R : A \otimes_p A \rightarrow A$, by $R(a \otimes b) = \phi(b)a$ and also let define bounded linear map $L : A \otimes_p A \rightarrow A$ by $L(a \otimes b) = \phi(a)b$ for every a and b in A . Moreover, one can easily verify that $\phi(R(x)) = \phi(L(x)) = \phi(\pi_A(x))$ for every $x \in A \otimes_p A$. Also, we have $L(x)a = L(x \cdot a)$, $L(a \cdot x) = \phi(a)L(x)$, $aR(x) = R(a \cdot x)$ and $R(x \cdot a) = \phi(a)R(x)$ for every $a \in A$ and $x \in A \otimes_p A$. Using the above considerations, we have

$$\|\phi(a)L(m_\alpha) - L(m_\alpha)a\| = \|L(a \cdot m_\alpha) - L(m_\alpha \cdot a)\| \leq \|L\| \|a \cdot m_\alpha - m_\alpha \cdot a\| \rightarrow 0.$$

Similarly it is easy to see that $aR(m_\alpha) - \phi(a)R(m_\alpha) \rightarrow 0$.

The nets $(R(m_\alpha))_\alpha$ and $(L(m_\alpha))_\alpha$ are denoted by $(R_\alpha)_\alpha$ and $(L_\alpha)_\alpha$, respectively. Suppose that the triple (E^{**}, P^{**}, g) is ϕ -admissible with respect to a ϕ -character ψ on P^{**} , so there exists a character ψ_0 on P such that its extension to P^{**} is ψ , hence there exists a bounded linear map $\eta : P^{**} \rightarrow E^{**}$ such that $\psi \circ g \circ \eta = \psi$. Let define $\rho : P^{**} \rightarrow E^{**}$ by $\rho(x) = w^* - \lim R_\alpha \cdot \eta(L_\alpha \cdot x \cdot R_\alpha) \cdot L_\alpha$. Since $(L_\alpha)_\alpha$ and $(R_\alpha)_\alpha$ are bounded nets and E^{**} is a dual Banach A -module, this definition makes sense. Using the boundedness of $(R_\alpha)_\alpha$ and $(L_\alpha)_\alpha$ and since $L_\alpha a - \phi(a)L_\alpha \rightarrow 0$, one can show that

$$R_\alpha \cdot \eta(L_\alpha a \cdot x \cdot R_\alpha) \cdot L_\alpha - \phi(a)R_\alpha \cdot \eta(L_\alpha \cdot x \cdot R_\alpha) \cdot L_\alpha \xrightarrow{\|\cdot\|} 0,$$

therefore

$$R_\alpha \cdot \eta(L_\alpha a \cdot x \cdot R_\alpha) \cdot L_\alpha - \phi(a) R_\alpha \cdot \eta(L_\alpha \cdot x \cdot R_\alpha) \cdot L_\alpha \xrightarrow{w^*} 0.$$

Similarly, we have

$$\phi(a) R_\alpha \cdot \eta(L_\alpha \cdot x \cdot R_\alpha) \cdot L_\alpha - a R_\alpha \cdot \eta(L_\alpha \cdot x \cdot R_\alpha) \cdot L_\alpha \xrightarrow{w^*} 0.$$

Hence,

$$\rho(a \cdot x) = a \cdot w^* - \lim R_\alpha \cdot \eta(L_\alpha \cdot x \cdot R_\alpha) \cdot L_\alpha = a \cdot \rho(x).$$

Thus ρ is a bounded left A -module morphism. Similarly, one can show that ρ is a bounded right A -module morphism. To finish the proof, we have to show that $\psi \circ g \circ \rho = \psi$. To see this, let $x \in P^{**}$. Using the $w^* - w^*$ continuity of the bounded A -bimodule morphism g and since $\psi = \tilde{\psi}_0$ is a ϕ -character and P^{**} is a dual Banach A -bimodule, we have

$$\begin{aligned} \psi \circ g \circ \rho(x) &= \psi \circ g(w^* - \lim R_\alpha \cdot \eta(L_\alpha \cdot x \cdot R_\alpha) \cdot L_\alpha) \\ &= \psi(w^* - \lim g(R_\alpha \cdot \eta(L_\alpha \cdot x \cdot R_\alpha) \cdot L_\alpha)) \\ &= \tilde{\psi}_0(w^* - \lim g(R_\alpha \cdot \eta(L_\alpha \cdot x \cdot R_\alpha) \cdot L_\alpha)) \\ &= w^* - \lim g(R_\alpha \cdot \eta(L_\alpha \cdot x \cdot R_\alpha) \cdot L_\alpha)(\psi_0) \\ &= \lim g(R_\alpha \cdot \eta(L_\alpha \cdot x \cdot R_\alpha) \cdot L_\alpha)(\psi_0) \\ &= \lim \psi(g(R_\alpha \cdot \eta(L_\alpha \cdot x \cdot R_\alpha) \cdot L_\alpha)) \\ &= \lim \psi(R_\alpha \cdot g \circ \eta(L_\alpha \cdot x \cdot R_\alpha) \cdot L_\alpha) \\ &= \lim \phi(R_\alpha) \phi(L_\alpha) \psi \circ g \circ \eta(L_\alpha \cdot x \cdot R_\alpha) \\ &= \lim \psi \circ g \circ \eta(L_\alpha \cdot x \cdot R_\alpha) \\ &= \lim \psi(L_\alpha \cdot x \cdot R_\alpha) \\ &= \lim \phi(L_\alpha) \psi(x) \phi(R_\alpha) \\ &= \psi(x), \end{aligned}$$

because $\phi(R_\alpha) = \phi(L_\alpha) \rightarrow 1$. Then the proof is complete. \square

Corollary 3.13. *Let G be a locally compact group. Then G is amenable if and only if every ϕ -admissible, triple (E^{**}, P^{**}, g) is ϕ -split triple, where g is $w^* - w^*$ -continuous $L^1(G)$ -bimodule morphism and P, E are Banach $L^1(G)$ -bimodules and also $\phi \in \Delta(L^1(G))$.*

Proof. Let G be an amenable locally compact group. Then $L^1(G)$ is an amenable Banach algebra. Hence, there exists an element $m \in (L^1(G) \otimes_p L^1(G))^{**}$ such that $a \cdot m = m \cdot a$ and $\pi_{L^1(G)}^{**}(m)a = a$ for every $a \in L^1(G)$. Therefore it is easy to see that $\tilde{\phi} \circ \pi_{L^1(G)}^{**}(m) = 1$. So $L^1(G)$ is ϕ -Johnson amenable, therefore by previous Proposition, if part is complete.

For converse, consider the triple $((L^1(G) \otimes_p L^1(G))^{**}, L^1(G)^{**}, \pi_{L^1(G)}^{**})$ where $\pi_{L^1(G)}^{**}$ is a $w^* - w^*$ -continuous $L^1(G)$ -bimodule morphism. Let $\phi \in \Delta(L^1(G))$. Pick an element a_0 in $L^1(G)$ such that $\phi(a_0) = 1$. Define $\eta : L^1(G) \rightarrow L^1(G) \otimes_p L^1(G)$ by $\eta(a) = a \otimes a_0$, where $a \in L^1(G)$. Clearly

$L^1(G)^{**}$ is Banach $L^1(G)$ -bimodule and $\tilde{\phi}$ is a ϕ -character. It is easy to see that $\tilde{\phi} \circ \pi_{L^1(G)}^{**} \circ \eta^{**}(a) = \tilde{\phi}(a)$ for every $a \in L^1(G)^{**}$. Hence the triple $((L^1(G) \otimes_p L^1(G))^{**}, L^1(G)^{**}, \pi_{L^1(G)}^{**})$ is ϕ -admissible. Then is ϕ -split, so there exists an $L^1(G)$ -bimodule morphism $\rho : L^1(G)^{**} \rightarrow (L^1(G) \otimes_p L^1(G))^{**}$ such that $\tilde{\phi} \circ \pi_{L^1(G)}^{**} \circ \rho(a) = \tilde{\phi}(a)$, for every $a \in L^1(G)^{**}$. Hence $\tilde{\phi} \circ \pi_{L^1(G)}^{**} \circ \rho(a) = \phi(a)$, where $a \in L^1(G)$. Then $L^1(G)$ is a ϕ -biflat Banach algebra. Now apply [9, Lemma 4.1] to show that G is amenable. \square

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